Pergamon

Int. J. Solids Structures Vol. 32, No. 1, pp. 105–115, 1995

0020–7683/95 \$9.50 + .00

0020–7683(94)00101–4

THE GENERAL SOLUTION OF THREE-DIMENSIONAL PROBLEMS IN PIEZOELECTRIC MEDIA

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(*Received* 28 *September* 1993; *in revised form* 18 *April* 1994)

Abstract—The governing equations of three-dimensional problems in transversely isotropic piezoelectric material are composed of four second order partial differential equations, in which the displacement components and electric potential functions are the essential unknowns. In this paper the so-called general solutions of these equations are obtained with the aid of a set of new potential functions which are used to express the components of displacements and the electric potential functions. As an application of the general solution the problem of a half-space piezoelectrics acted by a concentrated lateral shear force is solved.

1. INTRODUCTION

Piezoelectric materials have been extensively used as transducers and sensors due to their intrinsic direct and converse piezoelectric effects that take place between electric fields and mechanical deformation, and they are playing a key role as active components in many branches of science and technology such as electronics, infranics, navigation and biology. For example, piezoelectric materials are acting as very important functional components in sonar projectors, fluid monitors, pulse generators and surface acoustic wave devices. Piezoelectric ceramic materials are used especially widely owing to their high piezoelectric performance, but the inherent weakness of piezoelectric ceramics is its brittleness in mechanical behavior. During operation severe mechanical stress occurs in piezoelectrics. The stress concentrations caused by mechanical or electric loads may lead to crack initiation and extension, and sometimes the stress concentrations may be high enough to fracture the parts. Meantime, piezoelectric ceramics usually have initial defects such as microcracks, microvoids, layer-separations and inclusions, which may also bring about the failures of components. To improve the performance and to predict the reliable service life of ceramic piezoelectric components, it is necessary to analyse theoretically and describe accurately the damage and fracture processes taking place in ceramic piezoelectric materials from the angle of coupled effects of mechanics and electrics. The study in fracture mechanics of piezoelectric ceramics has been paid more attention to in recent years. Deeg (1980) and Pak (1987) have addressed the plane and antiplane fracture problems of piezoelectric materials and obtained a close form solution of stress field and electric displacement near the crack tip. With the aid of the three-dimensional eigenfunction expansion method, Sosa and Pak (1990) have investigated the case in which the crack front is assumed to be straight and it is located along the transversely isotropic axis of symmetry and they have discussed the influence of electric fields to the stress field near the crack tip. Moreover, Sosa (1990) has suggested a general method of solving plane problems of piezoelectric media with defects. Usually cracks in piezoelectric media are of penny shape or elliptic shape, so that an effective method to solve the three-dimensional problem of piezoelectric media is necessary both in theoretical and practical interest. Wang (1992) has obtained the general solutions of governing equations to three-dimensional axisymmetric problems in transversely isotropic piezoelectric media and solved the problem of penny shaped cracks under stretching loads. In this paper the general solution of governing equations to three-dimensional problems in a transversely isotropic piezoelectric medium is obtained. The general solution is a powerful tool to solve problems such as void, inclusion and three-dimensional cracks in piezoelectric media.

2. BASIC EQUATIONS

The governing equations of the three-dimensional piezoelectricity in the absence of body forces and free charges can be written in compact form as follows:

$$\begin{aligned}
&\sigma_{ij,j} = 0,\\
&D_{i,i} = 0,\\
&\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k,\\
&D_i = e_{ikl}\varepsilon_{kl} + \epsilon_{ik}E_k,\\
&\varepsilon_{ij} = \tfrac{1}{2}(u_{j,i}+u_{i,j}),\\
&E_i = -\varphi_{,i}, \quad i,j,k,l = 1,2,3
\end{aligned}$$

where σ_{ij}, ε_{ij}, D_i, u_i, E_i are the components, respectively, of stress, strain, electric displacement, mechanical displacement and electric field, φ is the electric pontential function, and c_{ijke}, ϵ_{ij} and e_{kij} are the elastic stiffness constants, the dielectric constants are the piezoelectric constants, respectively. In the most general case of anisotropy, the piezoelectric material has 45 independent constants which are 21 elastic, 6 dielectric and 18 piezoelectric constants. In a transversely isotropic piezoelectric medium, there are 10 independent constants, namely, 5 elastic, 2 dielectric and 3 piezoelectric constants, to determine its mechanical and electric characteristics. In the latter case, a Cartesian system (x, y, z) is introduced and the z-axis is perpendicular to the isotropic plane of medium. We have

$$\left.\begin{aligned}
&\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = 0,\\
&\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = 0,\\
&\frac{\partial\tau_{zx}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_z}{\partial z} = 0,
\end{aligned}\right\} \tag{1}$$

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 0, \tag{2}$$

$$\left.\begin{aligned}
\sigma_x &= c_{11}\frac{\partial u}{\partial x} + c_{12}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial\varphi}{\partial z},\\
\sigma_y &= c_{12}\frac{\partial u}{\partial x} + c_{11}\frac{\partial v}{\partial y} + c_{13}\frac{\partial w}{\partial z} + e_{31}\frac{\partial\varphi}{\partial z},\\
\sigma_z &= c_{13}\frac{\partial u}{\partial x} + c_{13}\frac{\partial v}{\partial y} + c_{33}\frac{\partial w}{\partial z} + e_{33}\frac{\partial\varphi}{\partial z},\\
\tau_{yz} &= c_{44}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + e_{15}\frac{\partial\varphi}{\partial y},\\
\tau_{zx} &= c_{44}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + e_{15}\frac{\partial\varphi}{\partial x},\\
\tau_{xy} &= \frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right),
\end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} D_x &= e_{15}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)-\epsilon_{11}\frac{\partial \varphi}{\partial x}, \\ D_y &= e_{15}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)-\epsilon_{11}\frac{\partial \varphi}{\partial y}, \\ D_z &= e_{31}\frac{\partial u}{\partial x}+e_{31}\frac{\partial v}{\partial y}+e_{33}\frac{\partial w}{\partial z}-\epsilon_{33}\frac{\partial \varphi}{\partial z}. \end{aligned}\right\} \tag{4}$$

Equations (1) and (2) are equilibrium equations. The piezoelectric stress constitutive relationship is represented by eqns (3) and (4) in which the components of stress and electric displacement are expressed by displacement and electric potential functions. Substituting eqns (3) and (4) into (1) and (2), we have the governing differential equations as follows:

$$\left.\begin{aligned} & c_{11}\frac{\partial^2 u}{\partial x^2}+\frac{1}{2}(c_{11}-c_{12})\frac{\partial^2 u}{\partial y^2}+c_{44}\frac{\partial^2 u}{\partial z^2}+\frac{1}{2}(c_{11}+c_{12})\frac{\partial^2 v}{\partial x\partial y} \\ & \qquad +(c_{13}+c_{44})\frac{\partial^2 w}{\partial x\partial z}+(e_{15}+e_{31})\frac{\partial^2 \varphi}{\partial x\partial z}=0 \\ & \frac{1}{2}(c_{11}-c_{12})\frac{\partial^2 v}{\partial x^2}+c_{11}\frac{\partial^2 v}{\partial y^2}+c_{44}\frac{\partial^2 v}{\partial z^2}+\frac{1}{2}(c_{11}+c_{12})\frac{\partial^2 u}{\partial x\partial y} \\ & \qquad +(c_{13}+c_{44})\frac{\partial^2 w}{\partial y\partial z}+(e_{15}+e_{31})\frac{\partial^2 \varphi}{\partial y\partial z}=0 \\ & c_{44}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+c_{33}\frac{\partial^2 w}{\partial z^2}+(c_{13}+c_{44})\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}\right) \\ & \qquad +e_{15}\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}\right)+e_{33}\frac{\partial^2 \varphi}{\partial z^2}=0 \\ & e_{15}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)+e_{33}\frac{\partial^2 w}{\partial z^2}+(e_{15}+e_{31})\left(\frac{\partial^2 u}{\partial x\partial z}+\frac{\partial^2 v}{\partial y\partial z}\right) \\ & \qquad -\epsilon_{11}\left(\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}\right)-\epsilon_{33}\frac{\partial^2 \varphi}{\partial z^2}=0 \end{aligned}\right\} \tag{5}$$

where $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ are the displacement components and $\varphi(x,y,z)$ is the electric potential function. Equations (5) are the governing equations of the three-dimensional problems in a transversely isotropic piezoelectric medium.

3. GENERAL SOLUTION OF GOVERNING DIFFERENTIAL EQUATIONS

It seems to be extremely difficult to find the solution by means of direct integration due to the complexity of eqns (5). But the problem may become more tractable if we introduce a set of potential functions which could transform eqns (5) into the familiar differential equations, as follows

$$u=\frac{\partial \psi}{\partial x}-\frac{\partial \chi}{\partial y},\quad v=\frac{\partial \psi}{\partial y}+\frac{\partial \chi}{\partial x},\quad w=k_1\frac{\partial \psi}{\partial z},\quad \varphi=k_2\frac{\partial \psi}{\partial z}, \tag{6}$$

where, $\psi(x,y,z)$ and $\chi(x,y,z)$ are the potential functions introduced, and k_1 and k_2 are unknown constants. Substituting eqns (6) into (5), we have the following equations:

$$\frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2\chi}{\partial x^2}+\frac{\partial^2\chi}{\partial y^2}\right)+c_{44}\frac{\partial^2\chi}{\partial z^2}=0, \tag{7}$$

$$\left.\begin{aligned}
&c_{11}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+[c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2]\frac{\partial^2\psi}{\partial z^2}=0,\\
&[(c_{13}+c_{44})+c_{44}k_1+e_{15}k_2]\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+(c_{33}k_1+e_{33}k_2)\frac{\partial^2\psi}{\partial z^2}=0,\\
&[(e_{15}+e_{31})+e_{33}k_1-\epsilon_{11}k_2]\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)+(e_{33}k_1-\epsilon_{33}k_2)\frac{\partial^2\psi}{\partial z^2}=0.
\end{aligned}\right\} \tag{8}$$

To three-dimensional piezoelectric problems, the terms $(\partial^2\psi/\partial x^2)+(\partial^2\psi/\partial y^2)$ and $(\partial^2\psi/\partial z^2)$ are not identically equal to zero. Under this condition a nontrivial solution of eqns (8) is to exist only if they are identical equations, namely,

$$\left.\begin{aligned}
\frac{c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2}{c_{11}}&=\frac{c_{33}k_1+e_{33}k_2}{(c_{13}+c_{44})+c_{44}k_1+e_{15}k_2}=\lambda,\\
\frac{c_{44}+(c_{13}+c_{44})k_1+(e_{15}+e_{31})k_2}{c_{11}}&=\frac{e_{33}k_1-\epsilon_{33}k_2}{(e_{15}+e_{31})+e_{15}k_1-\epsilon_{11}k_2}=\lambda.
\end{aligned}\right\} \tag{9}$$

Eliminating k_1 and k_2 in above equations, we obtain a cubic algebra equation of λ:

$$A\lambda^3+B\lambda^2+C\lambda+D=0, \tag{10}$$

where

$$\begin{aligned}
A&=e_{15}^2+c_{44}\epsilon_{11},\\
B&=(2e_{15}^2c_{13}-c_{44}e_{31}^2+2e_{15}e_{31}c_{13}-2e_{15}c_{11}e_{33}+\epsilon_{11}c_{13}^2\\
&\quad+2c_{13}c_{44}\epsilon_{11}-c_{33}c_{11}\epsilon_{11}-c_{44}c_{11}\epsilon_{33})/c_{11},\\
C&=[(e_{15}+e_{31})^2c_{33}-2(c_{13}+c_{44})(e_{15}+e_{31})e_{33}\\
&\quad-\frac{e_{15}(e_{15}+e_{31})c_{44}c_{33}}{c_{13}+c_{44}}+\frac{e_{15}(e_{15}+e_{11})c_{44}c_{11}}{c_{13}+c_{44}}+\epsilon_{11}c_{44}c_{33}\\
&\quad+c_{11}e_{33}^2-\epsilon_{33}(c_{13}+c_{44})^2+\epsilon_{33}(c_{44}^2+c_{11}c_{33})]/c_{11},\\
D&=-(c_{44}e_{33}^2+\epsilon_{33}c_{11}c_{33})/c_{11}.
\end{aligned}$$

The three roots of eqn (10) are denoted by λ_j ($j=1, 2, 3$) and λ_1 is assumed to be a positive real number, λ_2 and λ_3 are either positive real numbers or a pair of conjugate complex roots with positive real parts. Corresponding to the three roots, there are three potential functions ψ_j ($j=1, 2, 3$) and each of them must satisfy one of the following equations, respectively,

$$\frac{\partial^2\psi_j}{\partial x^2}+\frac{\partial^2\psi_j}{\partial y^2}+\lambda_j\frac{\partial^2\psi_j}{\partial z^2}=0,\quad j=1,2,3. \tag{11}$$

Now define λ_4 as follows

$$\lambda_4=\frac{2c_{44}}{c_{11}-c_{12}}.$$

Equations (7) can be rewritten as

$$\frac{\partial^2 \chi}{\partial x^2}+\frac{\partial^2 \chi}{\partial y^2}+\lambda_4\frac{\partial^2 \chi}{\partial z^2}=0. \tag{12}$$

If we let

$$\lambda_i = 1/s_i^2, \quad z_i = s_i z, \quad i = 1,2,3,4, \tag{13}$$

then, eqns (11) and (12) become the following forms:

$$\frac{\partial^2 \psi_j}{\partial x^2}+\frac{\partial^2 \psi_j}{\partial y^2}+\frac{\partial^2 \psi_j}{\partial z_j^2}=0, \tag{14}$$

$$\frac{\partial^2 \chi}{\partial x^2}+\frac{\partial^2 \chi}{\partial y^2}+\frac{\partial^2 \chi}{\partial z_4^2}=0. \tag{15}$$

It is thus clear that potential functions $\psi_j(x, y, z)$ and $\chi(x, y, z_j)$ satisfy Laplace equation in the coordinate systems (x, y, z_i) (i = 1, 2, 3, 4).

Substituting the values of λ_j into the two expressions of eqn (9), we get k_{1j} and k_{2j} corresponding to λ_j, then eqns (6) become

$$\left.\begin{aligned} u &= \frac{\partial}{\partial x}(\psi_1+\psi_2+\psi_3)-\frac{\partial \chi}{\partial y},\\ v &= \frac{\partial}{\partial y}(\psi_1+\psi_2+\psi_3)+\frac{\partial \chi}{\partial x},\\ w &= k_{11}\frac{\partial \psi_1}{\partial z}+k_{12}\frac{\partial \psi_2}{\partial z}+k_{13}\frac{\partial \psi_3}{\partial z},\\ \varphi &= k_{21}\frac{\partial \psi_1}{\partial z}+k_{22}\frac{\partial \psi_2}{\partial z}+k_{23}\frac{\partial \psi_3}{\partial z}. \end{aligned}\right\} \tag{16}$$

These are the so-called general solutions of potential functions of eqns (5) where ψ_j and χ satisfy eqns (14) and (15), respectively.

With this solution it is convenient to analyse some three-dimensional problems of piezoelectricity in a system of cylindrical coordinates. With respect to this new coordinates system, eqns (16) become

$$\left.\begin{aligned} u_r &= \frac{\partial}{\partial r}(\psi_1+\psi_2+\psi_3)-\frac{1}{r}\frac{\partial \chi}{\partial \theta},\\ v_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}(\psi_1+\psi_2+\psi_3)+\frac{\partial \chi}{\partial r},\\ w_z &= k_{11}\frac{\partial \psi_1}{\partial z}+k_{12}\frac{\partial \psi_2}{\partial z}+k_{13}\frac{\partial \psi_3}{\partial z},\\ \varphi &= k_{21}\frac{\partial \psi_1}{\partial z}+k_{22}\frac{\partial \psi_2}{\partial z}+k_{23}\frac{\partial \psi_3}{\partial z}. \end{aligned}\right\} \tag{17}$$

Substituting (17) into (3) and (4), we have the expressions of components of stress and electric displacement,

$$\sigma_r = \left(c_{11}\frac{\partial^2}{\partial r^2}+c_{12}\frac{1}{r}\frac{\partial}{\partial r}+c_{12}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\psi_1+\psi_2+\psi_3)$$
$$+(c_{13}k_{1j}+e_{31}k_{2j})\frac{\partial^2\psi_j}{\partial z^2}+c_{11}\frac{1}{r^2}\frac{\partial\chi}{\partial\theta}+c_{12}\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial\theta},$$

$$\sigma_\theta = \left(c_{12}\frac{\partial^2}{\partial r^2}+c_{11}\frac{1}{r}\frac{\partial}{\partial r}+c_{11}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\psi_1+\psi_2+\psi_3)$$
$$+(c_{13}k_{1j}+e_{31}k_{2j})\frac{\partial^2\psi_j}{\partial z^2}+c_{12}\frac{1}{r^2}\frac{\partial\chi}{\partial\theta}+c_{11}\frac{1}{r}\frac{\partial^2\chi}{\partial r\partial\theta},$$

$$\sigma_z = c_{13}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\psi_1+\psi_2+\psi_3)+(c_{33}k_{1j}+e_{33}k_{2j})\frac{\partial^2\psi_j}{\partial z^2},$$

$$\tau_{z\theta} = (c_{44}+c_{44}k_{1j}+e_{15}k_{2j})\frac{1}{r}\frac{\partial^2\psi_j}{\partial z\partial\theta}+c_{44}\frac{\partial^2\chi}{\partial r\partial z},$$

$$\tau_{zr} = (c_{44}+c_{44}k_{1j}+e_{15}k_{2j})\frac{\partial^2\psi_j}{\partial r\partial z}-c_{44}\frac{1}{r}\frac{\partial^2\chi}{\partial\theta\partial z},$$

$$\tau_{r\theta} = (c_{11}-c_{12})\left(\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial}{\partial\theta}\right)(\psi_1+\psi_2+\psi_3)+\frac{1}{2}(c_{11}-c_{12})\left(\frac{\partial^2\chi}{\partial r^2}-\frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}\right), \quad (18)$$

$$D_r = (e_{15}+e_{15}k_{1j}-\epsilon_{11}k_{2j})\frac{\partial^2\psi_j}{\partial r\partial z}-e_{15}\frac{1}{r}\frac{\partial^2\chi}{\partial\theta\partial z},$$

$$D_\theta (e_{15}+e_{15}k_{1j}-\epsilon_{11}k_{2j})\frac{1}{r}\frac{\partial^2\psi_j}{\partial\theta\partial z}+e_{15}\frac{\partial^2\chi}{\partial r\partial z},$$

$$D_z = e_{31}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(\psi_1+\psi_2+\psi_3)+(e_{33}k_{1j}-\epsilon_{33}k_{2j})\frac{\partial^2\psi_j}{\partial z^2}, \quad (19)$$

where $j = 1, 2, 3$ and summation convention with respect to the repeated index is implied.

It is extremely difficult to theoretically find the roots due to the complexities of expressions of each coefficient in eqns (10). In the above discussion, we put a limit to the values of λ_j, which is confirmed by the solution to a practical problem of ceramic piezoelectrics. For example, the material properties of a ceramic piezoelectrics PZT—6B are presented in Table 1. λ_j and s_j can be determined by eqns (10) and (13),

$$\lambda_1 = 3.92, \quad \lambda_2 = 0.73+0.87i, \quad \lambda_3 = 0.73-0.87i,$$
$$s_1 = 0.51, \quad s_2 = 1.02-0.48i, \quad s_3 = 1.02+0.48i.$$

It can be found that if λ_2 and λ_3 is a pair of conjugate complex numbers, then k_{12} and k_{13}, k_{22} and k_{23} are complex conjugates, and ψ_2 and ψ_3 is a pair of complex conjugate functions. Therefore, all of the displacement, electric potential, stress and electric displacement components determined by eqns (17)–(19) are real numbers.

Table 1. Material properties of a piezoelectric ceramic PZT—6B

Elastic stiffness (10^{10} Nm^{-2})					Piezoelectric coefficients (Cm^{-2})			Dielectric constants (10^{-10} Fm^{-1})	
c_{11}	c_{33}	c_{44}	c_{12}	c_{13}	e_{31}	e_{33}	e_{15}	ϵ_{11}	ϵ_{33}
16.8	16.3	2.71	6.0	6.0	−0.9	7.1	4.6	36	34

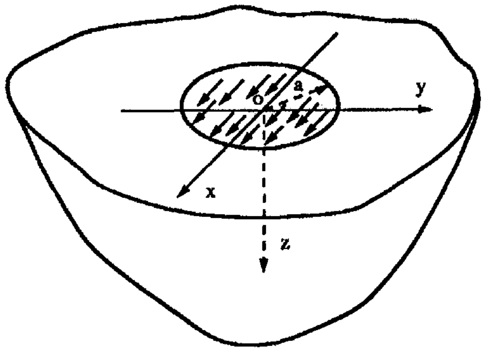

Fig. 1. The distributed shear force $p = P/\pi a^2$ acting on a circular plane.

4. A PRACTICAL EXAMPLE

To illustrate the application of the above general solution in engineering problems, a homogeneous half-space ceramic piezoelectrics is investigated in this section. Suppose that its surface coincides with the isotropic plane of the medium, and an in-plane concentrated force P acts at the point O, we now want to solve for the components of stress and displacement. To solve this problem we first discuss a problem with distributed shear force $p = P/\pi a^2$ acting on a circular plane shown in Fig. 1. The boundary conditions corresponding to this problem can be expressed as:

$$\sigma_z|_{z=0} = 0, \quad D_z|_{z=0} = 0, \tag{20}$$

$$\tau_{zr}|_{z=0} = \tau_{zr}|_{z=0} = \tau_{z\theta}(r, \theta) \tag{21}$$

while $\sqrt{r^2+z^2} \to \infty$, there are

$$\begin{aligned} &\sigma_r = \sigma_\theta = \sigma_z = \tau_{zr} = \tau_{z\theta} = \tau_{r\theta} = 0, \\ &D_r = D_\theta = D_z = 0. \end{aligned} \tag{22}$$

With respect to cylindrical coordinates (r, θ, z), eqns (14) can be rewritten as

$$\frac{\partial^2 \psi_j}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_j}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_j}{\partial \theta^2} = -\frac{1}{s_j^2}\frac{\partial^2 \psi}{\partial z^2} \quad (j = 1, 2, 3).$$

The expressions of σ_z and D_z determined by eqns (18) and (19) can be written as

$$\left.\begin{aligned} \sigma_z &= a_1 \frac{\partial^2 \psi_1}{\partial z_1^2} + a_2 \frac{\partial^2 \psi_2}{\partial z_2^2} + a_3 \frac{\partial^2 \psi_3}{\partial z_3^2}, \\ D_z &= b_1 \frac{\partial^2 \psi_1}{\partial z_1^2} + b_2 \frac{\partial^2 \psi_2}{\partial z_2^2} + b_3 \frac{\partial^2 \psi_3}{\partial z_3^2}, \end{aligned}\right\} \tag{23}$$

where we adopt $z_j = s_j z$ and expressions of a_j, b_j are

$$\begin{aligned} a_j &= -c_{13} + (c_{33}k_{1j} + e_{33}k_{2j})s_j^2, \\ b_j &= -e_{31} + (e_{33}k_{1j} - \epsilon_{33}k_{2j})s_j^2. \end{aligned}$$

Substituting eqn (23) into (20) we have

$$\left[\frac{\partial^2 \psi_2}{\partial z_2^2}\right]_{z=0} = K_1\left[\frac{\partial^2 \psi_1}{\partial z_1^2}\right]_{z=0}, \quad \left[\frac{\partial^2 \psi_3}{\partial z_3^2}\right]_{z=0} = K_2\left[\frac{\partial^2 \psi_1}{\partial z_1^2}\right]_{z=0}, \tag{24}$$

where

$$K_1 = \frac{a_3 b_1 - a_1 b_3}{a_2 b_3 - a_3 b_2}, \quad K_2 = \frac{a_2 b_1 - a_1 b_2}{a_2 b_3 - a_3 b_2}.$$

Equations (24) give the relationships between the potential functions ψ_j ($j = 1, 2, 3, 4$). According to the above procedure, we suppose that functions $\psi_j(x, y, z_j)$ and $\chi(x, y, z_4)$ have the following forms:

$$\left.\begin{aligned} \psi_1(r,\theta,z_1) &= f_1(r,\theta,z_1), \\ \psi_2(r,\theta,z_2) &= K_1 f_1(r,\theta,z_2) \\ \psi_3(r,\theta,z_3) &= K_2 f_1(r,\theta,z_3), \\ \chi(r,\theta,z_4) &= f_2(r,\theta,z_4), \end{aligned}\right\} \tag{25}$$

where functions $f_1(r,\theta,z_j)$ and $f_2(r,\theta,z_4)$ satisfy the Laplace equation in cylindrical coordinate system. By using Hankel integral transforms, functions $f_1(r,\theta,z_j)$ and $f_2(r,\theta,z_4)$ can be expressed in the following integral forms

$$\begin{aligned} \mu f_1(r,\theta,z_j) &= \sum_{m=0}^{\infty} \cos m\theta \int_0^{\infty} \frac{1}{\xi}[L_m(\xi)+I_m(\xi)] J_m(r\xi) e^{-\xi z_j}\,\mathrm{d}\xi \\ &\quad + \sum_{m=0}^{\infty} \sin m\theta \int_0^{\infty} \frac{1}{\xi}[D_m(\xi)+F_m(\xi)] J_m(r\xi) e^{-\xi z_j}\,\mathrm{d}\xi, \\ c_{44}s_4 f_2(r,\theta,z_4) &= -\sum_{m=0}^{\infty} \cos m\theta \int_0^{\infty} \frac{1}{\xi}[D_m(\xi)-F_m(\xi)] J_m(r\xi) e^{-\xi z_4}\,\mathrm{d}\xi \\ &\quad + \sum_{m=0}^{\infty} \sin m\theta \int_0^{\infty} \frac{1}{\xi}[L_m(\xi)-I_m(\xi)] J_m(r\xi) e^{-\xi z_4}\,\mathrm{d}\xi, \end{aligned} \tag{26}$$

where $J_m(r\xi)$ denotes the Bessel function of the first kind of order m, ξ is the Hankel transform parameter and $I_m(\xi)$, $L_m(\xi)$, $D_m(\xi)$ and $F_m(\xi)$ are arbitrary functions to be determined by using the given boundary conditions. The value of constant μ is

$$\mu = (c_{44}+c_{44}k_{11}+e_{15}k_{21})s_1 + K_1(c_{44}k_{12}+e_{15}k_{22})s_2 + K_2(c_{44}+c_{44}k_{13}+e_{15}k_{23})s_3.$$

Considering eqns (25), (26) and (18) and $z_j = 0$, we have

$$\begin{aligned} \tau_{zr}|_{z=0} &= -\sum_{m=0}^{\infty} \cos m\theta \int_0^{\infty} \xi[I_m(\xi)J_{m-1}(r\xi) - I_m(\xi)J_{m+1}(r\xi)]\,\mathrm{d}\xi \\ &\quad - \sum_{m=0}^{\infty} \sin m\theta \int_0^{\infty} \xi[D_m(\xi)J_{m-1}(r\xi) - F_m(\xi)J_{m+1}(r\xi)]\,\mathrm{d}\xi, \\ \tau_{z\theta}|_{z=0} &= -\sum_{m=0}^{\infty} \cos m\theta \int_0^{\infty} \xi[D_m(\xi)J_{m-1}(r\xi) + F_m(\xi)J_{m+1}(r\xi)]\,\mathrm{d}\xi \end{aligned}$$

$$+\sum_{m=0}^{\infty} \sin m\theta \int_0^{\infty} \xi[I_m(\xi)J_{m-1}(r\xi)+L_m(\xi)J_{m+1}(r\xi)]\,\mathrm{d}\xi. \tag{27}$$

Meanwhile, eqns (21) can be written as

$$\left.\begin{aligned} \tau_{zr}|_{r=0} &= \sum_{m=0}^{\infty} [p_m(r)\cos m\theta + q_m(r)\sin m\theta], \\ \tau_{z\theta}|_{z=0} &= \sum_{m=0}^{\infty} [s_m(r)\cos m\theta + t_m(r)\sin m\theta], \end{aligned}\right\} \tag{28}$$

where $p_m(r)$, $q_m(r)$, $s_m(r)$ and $t_m(r)$ are known functions prescribed on the surface of the half-space medium. Putting the two right-hand side terms in eqns (27) and (28) to be equal and using Hankel inversion transforms, we obtain

$$\left.\begin{aligned} I_m(\xi) &= \frac{1}{2}\int_0^{\infty} r[-p_m(r)+t_m(r)]J_{m-1}(r\xi)\,\mathrm{d}r, \\ L_m(\xi) &= \frac{1}{2}\int_0^{\infty} r[p_m(r)+t_m(r)]J_{m+1}(r\xi)\,\mathrm{d}r, \\ D_m(\xi) &= \frac{1}{2}\int_0^{\infty} r[-q_m(r)-s_m(r)]J_{m-1}(r\xi)\,\mathrm{d}r, \\ F_m(\xi) &= \frac{1}{2}\int_0^{\infty} r[q_m(r)-s_m(r)]J_{m+1}(r\xi)\,\mathrm{d}r. \end{aligned}\right\} \tag{29}$$

The distributed forces shown Fig. 1 can be expressed as

$$\tau_{zr}|_{z=0} = -\frac{P}{\pi a^2}\cos\theta, \quad \tau_{z\theta}|_{z=0} = \frac{P}{\pi a^2}\sin\theta \quad (0 \leqslant r \leqslant a).$$

It is clear that in all of $p_m(r)$, $q_m(r)$, $s_m(r)$ and $t_m(r)$, $m = 0, 1, 2, \ldots$, $p_1(r)$ and $t_1(r)$ are not equal to zero namely

$$p_1(r) = -\frac{P}{\pi a^2}, \quad t_1(r) = \frac{P}{\pi a^2}.$$

From eqns (29), we also find that only $I_1(\xi)$ is not equal to zero among $I_m(\xi)$, $L_m(\xi)$, $D_m(\xi)$ and $F_m(\xi)$

$$I_1(\xi) = \frac{P}{\pi a}\frac{J_1(a\xi)}{\xi}. \tag{30}$$

Substituting eqn (30) into eqn (26), and the above result into eqn (25), we obtain

$$\left.\begin{aligned}
\psi_1(r,\theta,z_1) &= \frac{P}{\pi a\mu}\cos\theta\int_0^\infty \frac{1}{\xi^2}J_1(a\xi)J_1(r\xi)e^{-\xi z_1}\,\mathrm{d}\xi,\\
\psi_2(r,\theta,z_2) &= \frac{K_1P}{\pi a\mu}\cos\theta\int_0^\infty \frac{1}{\xi^2}J_1(a\xi)J_1(r\xi)e^{-\xi z_2}\,\mathrm{d}\xi,\\
\psi_3(r,\theta,z_3) &= \frac{K_2P}{\pi a\mu}\cos\theta\int_0^\infty \frac{1}{\xi^2}J_1(a\xi)J_1(r\xi)e^{-\xi z_3}\,\mathrm{d}\xi,\\
\chi(r,\theta,z_4) &= -\frac{P}{\pi a c_{44}s_4}\sin\theta\int_0^\infty \frac{1}{\xi^2}J_1(a\xi)J_1(r\xi)e^{-\xi z_4}\,\mathrm{d}\xi.
\end{aligned}\right\} \tag{31}$$

And lastly, substituting eqn (31) into eqn (17) and using the following expressions

$$\lim_{a\to 0}\frac{J_1(a\xi)}{a\xi} = \frac{1}{2}$$

$$\int_0^\infty \frac{1}{\xi^2}J_1(a\xi)J_1(r\xi)e^{-\xi z_i}\,\mathrm{d}\xi = \frac{a}{2}\frac{1}{r}[(r^2+z_i^2)^{1/2}-z_i],$$

$$\int_0^\infty \frac{1}{\xi}J_1(a\xi)J_0(r\xi)e^{-\xi z_i}\,\mathrm{d}\xi = \frac{a}{2}(r^2+z_i^2)^{-1/2},$$

$$\int_0^\infty \frac{1}{\xi}J_1(a\xi)J_1(r\xi)e^{-\xi z_i}\,\mathrm{d}\xi = \frac{a}{2}\frac{1}{r}[1-z_i(r^2+z_i^2)^{-1/2}],$$

$$\int_0^\infty J_1(a\xi)J_1(r\xi)e^{-\xi z_i}\,\mathrm{d}\xi = \frac{a}{2}r(r^2+z_i^2)^{-3/2},$$

$$\int_0^\infty J_1(a\xi)J_0(r\xi)e^{-\xi z_i}\,\mathrm{d}\xi = \frac{a}{2}z_i(r^2+z_i^2)^{-3/2},$$

we obtain

$$\begin{aligned}
u_r &= \frac{P}{2\pi\mu}\left(\frac{z_1R_1-z_1^2}{r^2R_1}+K_1\frac{z_2R_2-z_2^2}{r^2R_2}+K_2\frac{z_3R_3-z_3^2}{r^2R_3}\right)\cos\theta+\frac{P}{2\pi c_{44}s_4}\frac{1}{r^2}\left(1-\frac{z_4}{R_4}\right)\cos\theta,\\
v_\theta &= \frac{P}{2\pi\mu}\left(\frac{R_1+K_1R_2+K_2R_3}{r^2}-\frac{z_1+K_1z_2+K_2z_3}{r^2}\right)\sin\theta-\frac{P}{2\pi c_{44}s_4}\left(\frac{1}{R_4}-\frac{R_4-z_4}{r}\right)\sin\theta,\\
w &= -\frac{P}{2\pi\mu}\left(k_{11}s_1\frac{R_1-z_1}{r}+k_{12}K_1s_2\frac{R_2-z_2}{r}+k_{13}K_2s_3\frac{R_3-z_3}{r}\right)\cos\theta,\\
\varphi &= -\frac{P}{2\pi\mu}\left(k_{21}s_1\frac{R_1-z_1}{r}+k_{22}K_1s_2\frac{R_2-z_2}{r}+k_{23}K_2s_3\frac{R_3-z_3}{r}\right)\cos\theta,
\end{aligned} \tag{32}$$

where $R_i = \sqrt{r^2+z_i^2}$ (i = 1, 2, 3, 4). Making use of eqns (3), the components of stress and electric displacement can be obtained from eqns (18) and (19). As a result, the expressions of σ_r, τ_{zr}, $\tau_{z\theta}$ and D_z can be given as follows

$$\begin{aligned}
\sigma_r = &-\frac{P}{2\pi\mu}(c_{11}-c_{12})\frac{R_1+K_1R_2+K_2R_3}{r_3}\cos\theta+\frac{P}{2\pi\mu}(c_{11}-c_{12})\left(\frac{1}{R_1}+\frac{K_1}{R_2}+\frac{K_2}{R_3}\right)\frac{1}{r}\cos\theta\\
&+\frac{P}{2\pi\mu}c_{11}\left(\frac{1}{R_1^{3/2}}+\frac{K_1}{R_2^{3/2}}+\frac{K_2}{R_3^{3/2}}\right)\cos\theta-\frac{P}{2\pi\mu}\left[\frac{H_1s_1^2}{R_1^{3/2}}+\frac{H_2s_2^2K_1}{R_2^{3/2}}+\frac{H_3s_3^2K_2}{R_3^{3/2}}\right]r\cos\theta
\end{aligned}$$

$$-\frac{P}{2\pi c_{44}s_4}\left(c_{11}\frac{R_4-z_4}{r^3}+c_{12}\frac{1}{rR_4}-c_{12}\frac{R_4-z_4}{r^3}\right)\cos\theta,$$

$$\tau_{zr}=\frac{P}{2\pi\mu}\left[Q_1\left(\frac{z_1}{R_1^{3/2}}-\frac{R_1-z_1}{r^2R_1}\right)+K_1Q_2\left(\frac{z_2}{R_2^{3/2}}-\frac{R_2-z_2}{r^2R_2}\right)\right.$$

$$\left.+K_2Q_3\left(\frac{z_3}{R_3^{3/2}}-\frac{R_3-z_3}{r^2R_3}\right)\right]\cos\theta-\frac{P}{2\pi}\frac{R_4-z_4}{r^2R_4}\cos\theta,$$

$$\tau_{z\theta}=\frac{P}{2\pi\mu}\left[Q_1s_1\left(\frac{R_1-z_1}{r^2R_1}+Q_2K_1s_2\frac{R_2-z_2}{r^2R_2}+Q_3K_2s_3\frac{R_3-z_3}{r^2R_3}\right)\right]\sin\theta$$

$$-\frac{P}{2\pi}\left(\frac{z_4}{R_4^{3/2}}-\frac{R_4-z_4}{r^2R_4}\right)\sin\theta,$$

$$D_z=\frac{P}{2\pi\mu}\left(\frac{b_1}{R_1^{3/2}}+\frac{b_2K_1}{R_2^{3/2}}+\frac{b_3K_2}{R_3^{3/2}}\right)r\cos\theta,$$

$$\ldots\ldots \tag{33}$$

where

$$H_j=c_{13}k_{1j}+e_{31}k_{2j},$$

$$Q_j=c_{44}+c_{44}k_{1j}+e_{15}k_{2j}\quad(j=1,2,3).$$

5. CONCLUSIONS

In this paper, the general solution of three-dimensional problems of transversely isotropic ceramic media based on potential functions is obtained. Using the general solution, we can get the analytical expressions of stress and electric displacement in some cases of loading in piezoelectricity. Furthermore, the general solution provides a powerful tool for solving various three-dimensional crack problems in piezoelectric ceramic media.

Acknowledgement—This research was supported by the National Natural Science Foundation.

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